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q-Analogues of the Riemann zeta, the Dirichlet L-functions, and a crystal zeta function

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Abstract

A q-analogue $\zeta_q(s)$ of the Riemann zeta function $\zeta(s)$ was studied in [Kaneko et al. 03] via a certain q-series of two variables. We introduce in a similar way a q-analogue of the Dirichlet L-functions and make a detailed study of them, including some issues concerning the classical limit of $\zeta_q(s)$ left open in [Kaneko et al. 03]. We also examine a "crystal" limit (i.e. $q \downarrow 0$) behavior of $\zeta_q(s)$. The q-trajectories of the trivial and essential zeros of $\zeta(s)$ are investigated numerically when q moves in $(0,1]$. Moreover, conjectures for the crystal limit behavior of zeros of $\zeta_q(s)$ are given.
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1 Introduction

There are fairly plenty of possibilities for defining q-analogues of the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ in the convergent region $\mathrm{Re}(s)>1$, (See, e.g., [Satoh 89], [Cherednik 01] and [Kaneko et al. 03]). Among them, in [Kaneko et al. 03] Kaneko, Kurokawa and the second author introduced a certain q-analogue $\zeta_q(s)$ of $\zeta(s)$ which is meromorphically extended to the entire plane $\mathbb{C}$ and indeed gives a proper q-analogue in the sense that the classical limit of $\zeta_q(s)$ exists and equals $\zeta(s)$ for *all* $s\in\mathbb{C}$. We briefly recall the story. Suppose that $0<q<1$. Let $f_q(s,t)$ be a function of the two complex variables s and t defined by the series $f_q(s,t):=\sum_{n=1}^{\infty}q^{nt}[n]_q^{-s}$, where $[n]_q:=(1-q^n)/(1-q)$ denotes a q-analogue of the number n. It is clear that the series converges absolutely for $\mathrm{Re}(t)>0$. Obviously, $\lim_{q\uparrow 1}f_q(s,t)=\zeta(s)$ holds in the convergent region $\mathrm{Re}(t)>0$. Among these $f_q(s,t)$, the q-analogue $\zeta_q(s)$ of $\zeta(s)$ is defined by setting $\zeta_q(s):=f_q(s,s-1)$ for $\mathrm{Re}(s)>1$. Then it was shown that $\zeta_q(s)$ can be meromorphically continued to $\mathbb{C}$ and moreover that $\lim_{q\uparrow 1}\zeta_q(s)=\zeta(s)$ for all $s\in\mathbb{C}$. Note that, however, neither an Euler product expression nor a functional equation can be expected for $\zeta_q(s)$.

The initial aim of the present paper is to generalize the result on $\zeta_q(s)$ to the cases of the Dirichlet L-functions and to make a much detailed study of those q-analogues including some results about the case of the Riemann zeta function not even clarified in [Kaneko et al. 03]. Let $N\in\mathbb{N}$ and χ be a Dirichlet character modulo N. Let $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ ($\mathrm{Re}(s)>1$) be the Dirichlet

L-functions. To explain the result precisely, let us define a series $f_q(s,t,\chi)$ similar to $f_q(s,t)$ by the formula $f_q(s,t,\chi) = \sum_{n=1}^{\infty} \chi(n) q^{nt} [n]_q^{-s}$ for $\mathrm{Re}(t) > 0$. It is then proved that $f_q(s,t,\chi)$ can be meromorphically continued to the entire plane $\mathbb{C}$. Putting $L_q(s,\chi) = f_q(s,s-1,\chi)$, as one may expect from the study of $\zeta_q(s)$, we actually prove that $\lim_{q\uparrow 1} L_q(s,\chi) = L(s,\chi)$ for all $s \in \mathbb{C}$. Moreover, not only do we treat the case of $L_q(s,\chi)$, but also we show that each of the functions $f_q(s,s-\nu,\chi)$, $\nu = 2,3,\ldots$, gives a proper q-analogue of $L(s,\chi)$, and, what is more, that only these functions $f_q(s,s-\nu,\chi)$ ($\nu \in \mathbb{N}$) can realize such true q-analogues of $L(s,\chi)$ in the family of the meromorphic functions of the form $f_q(s,\varphi(s),\chi)$ provided $\varphi(s)$ is a non-constant meromorphic function on $\mathbb{C}$. If χ is not principal, however, in addition to the case $\varphi(s) = s-\nu$ for $\nu \in \mathbb{N}$, the constant function $\varphi(s) = \mu \in \mathbb{N}$ gives also a true q-analogue of $L(s,\chi)$. Our analysis is based on the use of a q-analogue of the Hurwitz zeta function defined similarly.

A numerical analysis of the zeros of $\zeta_q(s)$, which is the second purpose of this paper, is developed in the last section §3. We first examine a "crystal" limit (i.e. the pointwise limit for $q \downarrow 0$) of $\zeta_q(s)$ and show that an analogue of the Riemann hypothesis holds for such a crystal Riemann zeta function. Furthermore, the q-trajectories of the trivial zeros and the essential zeros of $\zeta(s)$ are numerically investigated, that is, the zeros of $\zeta_q(s)$ are studied with Maple 8 [Maple 03]. Especially, we observe that the limit point of the q-trajectory of each essential zero of $\zeta(s)$ falls on either 0 or the one of the trajectory of the some trivial zero of $\zeta(s)$, i.e. on a negative integer point. We then give some conjectures concerning the crystal limit behavior of zeros of $\zeta_q(s)$.

2 On classical limits

We study the q-analogues of the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ and the Hurwitz zeta function $\zeta(s,a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ defined respectively by the series

$$f_q(s,t,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n) q^{nt}}{[n]_q^s} \quad \text{and} \quad g_q(s,t,a) := \sum_{n=0}^{\infty} \frac{q^{(n+a)t}}{[n+a]_q^s},$$

where χ is a Dirichlet character modulo N and $0 < a \leq 1$. These series converge absolutely for $\mathrm{Re}(t) > 0$. Obviously, $f_q(s,t) = f_q(s,t,\mathbf{1}) = g_q(s,t,1)$ ($\mathbf{1}$ denotes the trivial character), where $f_q(s,t)$ is the function discussed in [Kaneko et al. 03] (See §1). We put $L_q^{(\nu)}(s,\chi) := f_q(s,s-\nu,\chi)$, $\zeta_q^{(\nu)}(s) := f_q(s,s-\nu,\mathbf{1})$ for $\nu \in \mathbb{N}$, and $\zeta_q(s) = \zeta_q^{(1)}(s)$. These series converge absolutely for $\mathrm{Re}(s) > \nu$. We will show later that if χ is not the principal character, $L_q^{(\nu)}(s,\chi)$ is holomorphic at $s = \nu, \nu-1, \nu-2, \ldots, 1$, and if χ is the principal character, $L_q^{(\nu)}(s,\chi)$ has simple poles at these points. We find, in particular, that $\zeta_q^{(\nu)}(s)$ has simple poles at these points (See Proposition 2.9). Further we define an entire function $L_q^{\mu}(s,\chi)$ for $\mu \in \mathbb{N}$ by

$$L_q^{\mu}(s,\chi) := f_q(s,\mu,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n) q^{\mu n}}{[n]_q^s}.$$

We first prove the following theorem, which, in particular, gives an affirmative answer of the question in [Kaneko et al. 03], Remark (1) on p. 179.

Theorem 2.1. *Let $t=\varphi(s)$ be a meromorphic function on $\mathbb{C}$. Then the formula*

$$\lim_{q\uparrow 1} g_q(s,\varphi(s),a)=\zeta(s,a) \quad (s\in\mathbb{C})$$

holds if and only if the function $\varphi(s)$ can be written as $\varphi(s)=s-\nu$ for some $\nu\in\mathbb{N}$. Namely, $g_q(s,s-\nu,a)$ gives a true q-analogue of $\zeta(s,a)$ and these are the only true q-analogues among the functions of the form $g_q(s,\varphi(s),\chi)$.

To prove Theorem 2.1, recall the Euler-Maclaurin summation formula (see, [Titchmarsh 86]). For integers b,c satisfying $b<c$, a C^∞-function $f(x)$ on $[b,\infty)$, and an arbitrary integer $M\geq 0$, we have

$$\sum_{n=b}^{c} f(n)=\int_b^c f(x)dx+\frac{1}{2}(f(b)+f(c))+\sum_{l=1}^{M}\frac{B_{l+1}}{(l+1)!}(f^{(l)}(c)-f^{(l)}(b)) - \frac{(-1)^{M+1}}{(M+1)!}\int_b^c \widetilde{B}_{M+1}(x)f^{(M+1)}(x)dx, \tag{2.1}$$

where B_n is the Bernoulli number and $\widetilde{B}_{M+1}(x)$ is the periodic Bernoulli polynomial defined by $\widetilde{B}_k(x)=B_k(x-[x])$ with $[x]$ being the largest integer not exceeding x. Assume $\mathrm{Re}(t)>0$. Since $g_q(s,t,a)=(1-q)^s\sum_{n=0}^{\infty}q^{(n+a)t}(1-q^{n+a})^{-s}$, the Euler-Maclaurin summation formula (2.1) gives

$$g_q(s,t,a)=\frac{1}{2}q^{at}\Big(\frac{1-q^a}{1-q}\Big)^{-s}-\frac{1}{12}q^{at}\Big(\frac{1-q^a}{1-q}\Big)^{-s}\left(\Big(\frac{1-q^a}{\log q}\Big)^{-1}s+(t-s)\log q\right) + (1-q)^s I_0^0(s,t;q)+(1-q)^s\sum_{\varepsilon=0}^{2}a_\varepsilon(s,t;q)I_\varepsilon(s,t;q),$$

where

$$\begin{aligned} I_0^0(s,t;q)&=\int_0^\infty q^{(x+a)t}(1-q^{x+a})^{-s}dx,\\ I_\varepsilon(s,t;q)&=\int_0^\infty \widetilde{B}_2(x)q^{(x+a)t}(1-q^{x+a})^{-s-\varepsilon}dx \quad (\varepsilon=0,1,2),\\ a_\varepsilon(s,t;q)&=\begin{cases} -\frac{1}{2}(\log q)^2(s-t)^2 & (\varepsilon=0),\\ \frac{1}{2}(\log q)^2 s(2s-2t+1) & (\varepsilon=1),\\ -\frac{1}{2}(\log q)^2 s(s+1) & (\varepsilon=2). \end{cases}\end{aligned}$$

Change the variables $u=q^{x+a}$ in the first integral $I_0^0(s,t;q)$. Then

$$I_0^0(s,t;q)=-\frac{1}{\log q}b_{q^a}(t,-s+1) \qquad \Big(\text{or} \quad b_{q^a}(\alpha,\beta)=-\log q\cdot I_0^0(-\beta+1,\alpha;q)\Big), \tag{2.2}$$

where $b_q(\alpha,\beta)$ is the incomplete beta function defined by

$$b_q(\alpha,\beta)=\int_0^q u^{\alpha-1}(1-u)^{\beta-1}du \qquad (0<q<1).$$

The integral $b_q(\alpha,\beta)$ converges absolutely for $\mathrm{Re}(\alpha)>0$. Moreover, the Fourier expansion

$$\widetilde{B}_k(x) = -k! \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \tag{2.3}$$

gives

$$I_\varepsilon(s,t;q) = \frac{2}{\log q} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{e^{-2\pi i n a}}{(2\pi i n)^2} b_{q^a}(t+\delta n, -s-\varepsilon+1), \tag{2.4}$$

where $\delta := 2\pi i/\log q$. Note that since $b_{q^a}(t+\delta n,-s-\varepsilon+1)$ converges absolutely for $\mathrm{Re}(t)>0$ and is uniformly bounded with respect to n, the sum (2.4) converges absolutely. Take $M\ge 2$ satisfying $\mathrm{Re}(s)>-M$. Then, integral by parts yields

$$\begin{aligned} b_q(\alpha,\beta) = &\sum_{l=1}^{M-1} (-1)^{l-1} \frac{(1-\beta)_{l-1}}{(\alpha)_l} q^{\alpha+l-1}(1-q)^{\beta-l} \\ &+ (-1)^{M-1}\frac{(1-\beta)_{M-1}}{(\alpha)_{M-1}} b_q(\alpha+M-1,\beta-M+1), \end{aligned} \tag{2.5}$$

where $(s)_k := s(s+1)(s+2)\cdots(s+k-1)$. Hence, by (2.5), we have

$$\begin{aligned} b_{q^a}(t+\delta n, -s-\varepsilon+1) = e^{2\pi i n a}\Bigg\{&(1-q^a)^{-s-\varepsilon+1}\sum_{k=1}^{M-1}\frac{(-1)^{k-1}(s+\varepsilon)_{k-1}}{(1-q^a)^k(t+\delta n)_k} q^{a(t+k-1)} \\ -\log q(1-q)^{-s-\varepsilon}&\frac{(-1)^{M-1}(s+\varepsilon)_{M-1}}{(1-q)^{M-1}(t+\delta n)_{M-1}} \int_0^\infty e^{2\pi i n x} q^{(x+a)(t+M-1)}\Big(\frac{1-q^{x+a}}{1-q}\Big)^{-s-\varepsilon+1-M} dx\Bigg\}. \end{aligned} \tag{2.6}$$

This gives an analytic continuation of $b_{q^a}(t+\delta n,-s-\varepsilon+1)$ to the region $\mathrm{Re}(t)>1-M$. Hence, by (2.6), we obtain

$$(1-q)^s a_\varepsilon(s,t;q) I_\varepsilon(s,t;q) = \frac{2(1-q^a)^{-\varepsilon+1}}{\log q} a_\varepsilon(s,t;q) C_\varepsilon(s,t,M;q) =: \widetilde{C}_\varepsilon(s,t,M;q). \tag{2.7}$$

Here we put

$$\begin{aligned} C_\varepsilon(s,t,M;q) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i n)^2}\Bigg\{&\Big(\frac{1-q^a}{1-q}\Big)^{-s}\sum_{k=1}^{M-1}\frac{(-1)^{k-1}(s+\varepsilon)_{k-1}}{(1-q^a)^k(t+\delta n)_k} q^{a(t+k-1)} \\ -\Big(\frac{1-q^a}{\log q}\Big)^{-1}\Big(\frac{1-q^a}{1-q}\Big)^{\varepsilon}&\frac{(-1)^{M-1}(s+\varepsilon)_{M-1}}{(1-q)^{M-1}(t+\delta n)_{M-1}}\int_0^\infty e^{2\pi i n x} q^{(x+a)(t+M-1)}\Big(\frac{1-q^{x+a}}{1-q}\Big)^{-s-\varepsilon+1-M} dx\Bigg\}. \end{aligned} \tag{2.8}$$

Lemma 2.2. *Let $M\ge 2$ be an integer satisfying $\mathrm{Re}(s)>-M$. Then $\lim_{q\uparrow 1}\widetilde{C}_\varepsilon(s,t,M;q)$ ($\varepsilon = 0,1,2$) exist and are given by*

$$\lim_{q\uparrow 1}\widetilde{C}_0(s,t,M;q) = \lim_{q\uparrow 1}\widetilde{C}_1(s,t,M;q) = 0,$$

$$\lim_{q\uparrow 1}\widetilde{C}_2(s,t,M;q) = \sum_{k=2}^{M}\frac{B_{k+1}}{(k+1)!}(s)_k a^{-s-k} - \frac{(s)_{M+1}}{(M+1)!}\int_0^\infty \widetilde{B}_{M+1}(x)(x+a)^{-s-1-M}dx.$$

Proof. The results follow from the fact $\lim_{q\uparrow 1}(1-q^a)^k(t+\delta n)_k = (-2\pi i n a)^k$, (2.4) and (2.8). $\square$

By (2.2), (2.4) and (2.7), we obtain the following proposition.

Proposition 2.3. *Let $M \geq 2$ be an integer satisfying* $\mathrm{Re}(s) > -M$. *Then we have*

$$g_q(s,t,a) = \frac{1}{2}q^{at}\Big(\frac{1-q^a}{1-q}\Big)^{-s} - \frac{1}{12}q^{at}\Big(\frac{1-q^a}{1-q}\Big)^{-s}\left(\Big(\frac{1-q^a}{\log q}\Big)^{-1}s + (t-s)\log q\right)$$
$$- \frac{(1-q)^s}{\log q}b_{q^a}(t,-s+1) + \sum_{\varepsilon=0}^{2}\widetilde{C}_\varepsilon(s,t,M;q).$$

This gives the analytic continuation of $g_q(s,t,a)$ to the region $\mathrm{Re}(t) > 1-M$. $\square$

Proof of Theorem 2.1. We first show the sufficiency, that is, for each $\nu \in \mathbb{N}$, $\lim_{q\uparrow 1} g_q(s,s-\nu,a) = \zeta(s,z)$ for all $s \in \mathbb{C}$. Initially, we assume $t = s-1$. Though the case is referred in [Kaneko et al. 03], Remark (1) on p. 184, we give a proof for completeness. The condition $\mathrm{Re}(t) > 0$ implies $\mathrm{Re}(s) > 1$. In this case, we can evaluate $b_{q^a}(s-1,-s+1)$ by an elementary function. In fact, in general, since

$$b_q(\alpha-\nu+1,-\alpha) = -\sum_{r=0}^{\nu-1}\frac{(-\nu+1)_r}{(-\alpha)_{r+1}}q^{\alpha-\nu+1}(1-q)^{-\alpha+r} \qquad (\nu \in \mathbb{N}, \mathrm{Re}(\alpha) > \nu-1), \tag{2.9}$$

we have $b_{q^a}(s-1,-s+1) = \frac{1}{s-1}q^{a(s-1)}(1-q^a)^{-s+1}$. Therefore, by Lemma 2.2, for $\mathrm{Re}(s) > 2-M$ we obtain

$$\lim_{q\uparrow 1} g_q(s,s-1,a) = \frac{1}{s-1}a^{1-s} + \frac{1}{2}a^{-s} + \sum_{k=1}^{M}\frac{B_{k+1}}{(k+1)!}(s)_k a^{-s-k}$$
$$- \frac{(s)_{M+1}}{(M+1)!}\int_0^\infty \widetilde{B}_{M+1}(x)(x+a)^{-s-1-M}dx. \tag{2.10}$$

Comparing the equation (2.10) with (the analytic continuation of) the Hurwitz zeta function

$$\zeta(s,a) = \frac{1}{s-1}a^{1-s} + \frac{1}{2}a^{-s} + \sum_{l=1}^{M}\frac{B_{l+1}}{(l+1)!}(s)_l a^{-s-l}$$
$$- \frac{(s)_{M+1}}{(M+1)!}\int_0^\infty \widetilde{B}_{M+1}(x)(x+a)^{-s-M-1}dx \qquad (\mathrm{Re}(s) > -M), \tag{2.11}$$

we have the claim for $\nu = 1$.

Suppose next $t = s-\nu$ for $\nu \geq 2$. Let $\mathcal{S}$ be an operator (essentially due to [Zhao]) defined as

$$\mathcal{S}g_q(s,s-\nu,a) := g_q(s,s-\nu,a) + (1-q)g_q(s-1,s-\nu-1,a).$$

Since it can be easily checked that $g_q(s,s-\nu-1,a) = \mathcal{S}g_q(s,s-\nu,a)$, we have inductively

$$g_q(s,s-\nu,a) = \mathcal{S}^{\nu-1}g_q(s,s-1,a) = \sum_{r=0}^{\nu-1}\binom{\nu-1}{r}(1-q)^r g_q(s-r,s-r-1,a). \tag{2.12}$$

Letting $q \uparrow 1$, we have $\lim_{q\uparrow 1} g_q(s, s-\nu, a) = \lim_{q\uparrow 1} g_q(s, s-1, a) = \zeta(s, a)$. Thus the claim follows.

We next show the necessity. Suppose that $\lim_{q\uparrow 1} g_q(s, \varphi(s), a)$ exists and $\lim_{q\uparrow 1} g_q(s, \varphi(s), a) = \zeta(s, a)$ for all $s \neq 1$. By Lemma 2.2 and Proposition 2.3, the limit $(1-q)^{s-1}\frac{(1-q)}{\log q} b_{q^a}(\varphi(s), -s+1)$ for $q \uparrow 1$ does exist for all $s \neq 1$. Assume $\mathrm{Re}(s) < 1$. Since $\lim_{q\uparrow 1}(1-q)^{s-1}$ diverges, in particular, it is necessary to have $\lim_{q\uparrow 1} b_{q^a}(\varphi(s), -s+1) = 0$. Since $\lim_{q\uparrow 1} b_{q^k}(\alpha, \beta) = B(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{C}$ with $B(\alpha, \beta)$ being the beta function, we see that $B(\varphi(s), -s+1) = \Gamma(\varphi(s))\Gamma(-s+1)/\Gamma(\varphi(s)-s+1) = 0$. Since the poles of the gamma function $\Gamma(s)$ are only at the non-positive integers, it follows that $\varphi(s) - s + 1 \in \mathbb{Z}_{\leq 0}$. This shows that $\varphi(s) = s - \nu$ for some $\nu \in \mathbb{N}$ for $\mathrm{Re}(s) < 1$. Since $\varphi(s)$ is meromorphic in $\mathbb{C}$, $\varphi(s) = s - \nu$ for all $s \in \mathbb{C}$. Hence the assertion follows. $\square$

By the discussion above, we may have infinitely many true q-analogues of the Hurwitz zeta function which are *not* of the form $f_q(s, \varphi(s), \chi)$ for some $\varphi(s)$. For instance, as a remark, we have the following corollary.

Corollary 2.4. *For a positive integer μ, let*

$$\begin{aligned}\zeta_q^{\mu}(s, a) :&= \frac{(\mu-1)!}{(1-s)_\mu}\frac{(1-q)^s}{\log q} + g_q(s, \mu, a)\\ &= \frac{(-1)^\mu(\mu-1)!}{(s-1)\cdots(s-\mu)}\frac{(1-q)^s}{\log q} + \sum_{n=0}^{\infty}\frac{q^{\mu(n+a)}}{[n+a]_q{}^s}.\end{aligned}$$

Then the function $\zeta_q^{\mu}(s, a)$ is a meromorphic function on $\mathbb{C}$ with simple poles at $s = 1, 2, \ldots, \mu$, and gives a true q-analogue of $\zeta(s, a)$; $\lim_{q\uparrow 1} \zeta_q^{\mu}(s, a) = \zeta(s, a)$ holds for all $s \in \mathbb{C}$ $(s \neq 1, 2, \ldots, \mu)$.

Proof. The first assertion is clear from the definition of $\zeta_q^{\mu}(s, a)$. Further, by Proposition 2.3 and Lemma 2.2, it is easy to observe the function $G_q(s, t, a)$ defined as

$$G_q(s, t, a) := g_q(s, t, a) + \frac{(1-q)^s}{\log q} b_{q^a}(t, -s+1) - h_q(s, t, a) \tag{2.13}$$

gives a true q-analogue of $\zeta(s, a)$ if $\lim_{q\uparrow 1} h_q(s, t, a) = \frac{1}{s-1}a^{-s+1}$. Since

$$b_{q^a}(\mu, -s+1) = \sum_{l=1}^{\mu}\frac{(-1)^{l-1}(1-\mu)_{l-1}}{(1-s)_l}q^{a(\mu-l)}(1-q^a)^{-s+l} + \frac{(-1)^\mu(\mu-1)!}{(s-1)\cdots(s-\mu)},$$

we have $\zeta_q^{\mu}(s, a) = G_q(s, \mu, a)$, where

$$h_q(s, t, z) = \sum_{l=1}^{\mu}\frac{(-1)^{l-1}(1-\mu)_{l-1}}{(1-s)_l}q^{a(\mu-l)}\left(\frac{1-q^a}{1-q}\right)^{-s+l}\frac{(1-q)^l}{\log q}.$$

Hence the claim follows immediately from the fact $\lim_{q\uparrow 1} h_q(s, t, a) = \frac{1}{s-1}a^{-s+1}$. $\square$

Remark 2.5. The function $\zeta_q^1(s, a)$ gives $\widetilde{\zeta}_q(s, a)$ in [Tsumura 01].

Theorem 2.6. *Let χ be a Dirichlet character modulo N. Let $t = \varphi(s)$ be a meromorphic function on $\mathbb{C}$. Then the formula*

$$\lim_{q\uparrow 1} f_q(s,\varphi(s),\chi) = L(s,\chi) \quad (s \in \mathbb{C})$$

holds if and only if the function $\varphi(s)$ can be written as

(i) $\varphi(s) = s - \nu$ ($\nu \in \mathbb{N}$) if χ is the principal character.

(ii) $\varphi(s) = s - \nu$ ($\nu \in \mathbb{N}$) or $\varphi(s) = \mu$ ($\mu \in \mathbb{N}$) if χ is not the principal character.

Proof. The sufficiency follows immediately from Theorem 2.1 and Corollary 2.4 by the formulas

$$f_q(s,t,\chi) = \frac{1}{[N]_q^s}\sum_{k=1}^{N}\chi(k)g_{q^N}(s,t,\frac{k}{N}) \quad \text{and} \quad L(s,\chi) = \frac{1}{N^s}\sum_{k=1}^{N}\chi(k)\zeta(s,\frac{k}{N}). \tag{2.14}$$

Therefore, it suffices to show the necessity. The claim (i) is obvious from Theorem 2.1. Thus we assume χ is not the principal. Suppose $\lim_{q\uparrow 1} f_q(s,t,\chi) = L(s,\chi)$ for all $s \in \mathbb{C}$. Then, by Proposition 2.3, (2.11) and (2.14), it holds that

$$-\lim_{q\uparrow 1}\frac{(1-q)^s}{N\log q}\sum_{k=1}^{N}\chi(k)b_{q^k}(t,-s+1) = \frac{1}{N}\frac{1}{s-1}\sum_{k=1}^{N}\chi(k)k^{1-s} \qquad (s\in\mathbb{C}). \tag{2.15}$$

Similarly to (2.5), we have for any integer $M \geq 2$

$$b_q(\alpha,\beta) = \sum_{l=1}^{M-1}\frac{(-1)^l(1-\alpha)_{l-1}}{(\beta)_l}q^{\alpha-l}(1-q)^{\beta+l-1} + \frac{(-1)^{M-1}(1-\alpha)_{M-1}}{(\beta)_{M-1}}b_q(\alpha-M+1,\beta+M-1)$$

when $\mathrm{Re}(\alpha) > M-1$. Thus we have

$$\begin{aligned} b_{q^k}(t,-s+1) = \frac{1}{s-1}q^{k(t-1)}(1-q^k)^{-s+1} + \sum_{l=2}^{M-1}\frac{(-1)^l(1-t)_{l-1}}{(-s+1)_l}q^{k(t-l)}(1-q^k)^{-s+l} \\ +\frac{(-1)^{M-1}(1-t)_{M-1}}{(-s+1)_{M-1}}b_{q^k}(t-M+1,-s+M). \end{aligned} \tag{2.16}$$

Hence, using (2.16), we find the formula (2.15) is equivalent to

$$(1-t)_{M-1}\lim_{q\uparrow 1}(1-q)^{s-1}\sum_{k=1}^{N}\chi(k)b_{q^k}(t-M+1,-s+M) = 0 \qquad (s\in\mathbb{C}) \tag{2.17}$$

for some $M \geq 2$. We divide the proof into the following two cases;

1. The case where $t = \varphi(s) = \mu \in \mathbb{N}$:

   Take M as $M = t+1$. By (2.5), we have

$$\begin{aligned} b_{q^k}(t-M+1,-s+M) = \frac{1}{t-(M-1)}q^{k(t-M+1)}(1-q^k)^{-s+M-1} \\ -\frac{s-(M-1)}{t-(M-1)}b_{q^k}(t-M+2,-s+M-1). \end{aligned}$$

Note that

$$b_{q^k}(1,-s+M-1)=\frac{1}{s-(M-1)}\big((1-q^k)^{-s+M-1}-1\big).$$

Hence we have $\big(t-(M-1)\big)b_{q^k}(t-M+1,-s+M)|_{t=M-1}=1$. Since $\sum_{k=1}^N\chi(k)=0$, the claim follows.

2. Otherwise:

It is clear that the condition (2.17) is equivalent to

$$\sum_{k=1}^N\chi(k)k^{1-s}\lim_{q\uparrow 1}(1-q^k)^{s-1}b_{q^k}(t-M+1,-s+M)=0$$

By (2.5), we see that for any integer $M'\geq 2$

$$\begin{aligned}(2.18)\qquad b_q(t-M+1,-s+M)&=B(t-M+1,-s+M)\\&+\sum_{l=1}^{M'-1}\frac{(-1)^{l-1}(1+s-M)_{l-1}}{(t-M+1)_l}q^{t-M+l}(1-q)^{-s+M-l}\\&-\frac{(-1)^{M'-1}(1+s-M)_{M'-1}}{(t-M+1)_{M'-1}}\int_q^1u^{t-M+M'-1}(1-u)^{-s+M-M'}du.\end{aligned}$$

Put, $M'=M-1$ in (2.18). Then we obtain

$$\begin{aligned}(2.19)\quad &\sum_{k=1}^N\chi(k)k^{1-s}(1-q^k)^{s-1}b_{q^k}(t-M+1,-s+M)\\&=B(t-M+1,-s+M)\sum_{k=1}^N\chi(k)k^{1-s}(1-q^k)^{s-1}\\&+\sum_{k=1}^N\chi(k)k^{1-s}\sum_{l=1}^{M-2}\frac{(-1)^{l-1}(1+s-M)_{l-1}}{(t-M+1)_l}q^{k(t-M+l)}(1-q^k)^{M-l-1}\\&-\frac{(-1)^{M-2}(1+s-M)_{M-2}}{(t-M+1)_{M-2}}\sum_{k=1}^N\chi(k)k^{1-s}\frac{(1-q^k)^s}{1-q^k}\int_{q^k}^1u^{t-2}(1-u)^{-s+1}du.\end{aligned}$$

Note that we have

$$\frac{1}{1-q^k}\int_{q^k}^1u^{t-2}(1-u)^{-s+1}du=p^{k(t-2)}(1-p^k)^{-s+1}$$

for some $q<p<1$ by the mean-value theorem. Hence we have $\lim_{q\uparrow 1}\frac{(1-q^k)^s}{1-q^k}\int_{q^k}^1u^{t-2}(1-u)^{-s+1}du=0$ for all $s\in\mathbb{C}$. Therefore, (2.17) is equivalent to

$$B(t-M+1,-s+M)\times\left\{\lim_{q\uparrow 1}(1-q)^{s-1}\sum_{k=1}^N\chi(k)k^{1-s}\Big(\frac{1-q^k}{1-q}\Big)^{s-1}\right\}=0\qquad(s\in\mathbb{C}).$$

Note that there exists always an integer $r \geq 0$ such that $\sum_{k=1}^{f} \chi(k)(k-1)^r \neq 0$. Assume $\mathrm{Re}(s) < 1 - r$. Since $\lim_{q \uparrow 1}(1-q)^{s-1} \sum_{k=1}^{N} \chi(k) k^{1-s} \big(\frac{1-q^k}{1-q}\big)^{s-1}$ diverges for such s by the following lemma, we conclude that $B(t-M+1, -s+M) = 0$. Hence the claim follows from the same discussion as in the proof of Theorem 2.1.

□

Lemma 2.7. *We have*

$$\Big(k^{-1}\frac{1-q^k}{1-q}\Big)^{s-1} = \sum_{i=0}^{r} (q-1)^i \binom{s-1}{i} \Big(\sum_{l=1}^{k-1} [l]_q\Big)^i k^{-i} + O((1-q)^{r+1}) \tag{2.20}$$

for $r \geq 0$. *In particular,* $\lim_{q\uparrow 1}\Big|(1-q)^{s-1}\sum_{k=1}^{N}\chi(k)k^{1-s}\big(\frac{1-q^k}{1-q}\big)^{s-1}\Big| = \infty$ *for* $\mathrm{Re}(s) < 1-r$ *provided* $\sum_{k=1}^{N} \chi(k)(k-1)^r \neq 0$.

Proof. Since $k^{-1}\frac{1-q^k}{1-q} = 1 + ([1]_q + [2]_q + \cdots + [k-1]_q)(q-1)k^{-1}$, (2.20) follows from the binomial expansion. The rest of the assertion follows immediately from (2.20). □

Remark 2.8. Let $f(z)$ be an arbitrary function of the form $f(z) = z^{-s+1}P(z) + z^{-1}Q(z^{-1})\delta(\chi)$ with $f(1) = 1$, $P(z)$ being a polynomial and $Q(z)$ being a holomorphic function at $z = 0$. Here $\delta(\chi) = 1$ if χ is not the principal character and $\delta(\chi) = 0$ otherwise. Then, it is clear from Theorem 2.6 that $\lim_{q\uparrow 1} L_f(s,\chi) = L(s,\chi)$ where $L_f(s,\chi) := \sum_{n=1}^{\infty} \chi(n) f(q^{-n})[n]_q^{-s}$. Note that, however, P can not be a (infinite) Taylor series because the series $L_f(s,\chi)$ does not converge.

Special values of $L_q^{(\nu)}(s,\chi)$ are obtained by the following expression (2.21) of $L_q^{(\nu)}(s,\chi)$, which gives also the information concerning the locations of the poles (and a meromorphic continuation again) of $L_q^{(\nu)}(s,\chi)$.

Proposition 2.9. *(i) Let χ be a Dirichlet character. Then $L_q^{(\nu)}(s,\chi)$ can be written as*

$$L_q^{(\nu)}(s,\chi) = (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} g_\chi(q^{s-\nu+r}), \tag{2.21}$$

where $g_\chi(q) = \sum_{k=1}^{N} \chi(k) q^k/(1-q^N)$. *In particular, if χ is not the principal character, then $L_q^{(\nu)}(s,\chi)$ is holomorphic at $s = 1, 2, \ldots, \nu$, and if χ is the principal character, then $L_q^{(\nu)}(s,\chi)$ has simple poles at these points. Especially, $\zeta_q^{(\nu)}(s)$ has simple poles at these points.*
(ii) Let m be a non-negative integer. Then we have

$$L_q^{(\nu)}(-m,\chi) = (1-q)^{-m}\left\{\sum_{r=0}^{m}(-1)^r\binom{m}{r} g_\chi(q^{-m+r-\nu}) + \frac{(-1)^{m+1} m!(\nu-1)!}{(m+\nu)!\log q} B_{0,\chi}\right\}, \tag{2.22}$$

where $B_{n,\chi}$ is the generalized Bernoulli number defined via $\sum_{k=1}^{N}\frac{\chi(k)te^{kt}}{e^{Nt}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}$.

Proof. Using the binomial theorem $(1-x)^{-s} = \sum_{r=0}^{\infty} \binom{s+r-1}{r} x^r$, we can show

$$f_q(s,t,\chi) = (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} g_\chi(q^{t+r}) \tag{2.23}$$

by the same way as in [Kaneko et al. 03]. Indeed, the change of the order of two summations can be justified because the series converge absolutely. We notice here that

$$g_\chi(q^{t+r}) = -\sum_{n=-1}^{\infty} B_{n+1,\chi} \frac{(\log q)^n}{(n+1)!} (t+r)^n. \tag{2.24}$$

By the help of (2.24), the equation (2.23) gives the rest of the first assertion. The formula (2.22) follows from (2.21). □

From (2.22) and (2.24) we have the well-known formula $L(-m,\chi) = -B_{m+1,\chi}/(m+1)$ for a non-negative integer m as the classical limit $q \uparrow 1$. We next consider the limit $q \downarrow 0$ of $L_q^{(\nu)}(s,\chi)$ which we call a *crystal limit*. We then introduce functions $L_0^{(\nu)}(s,\chi)$ and $\zeta_0^{(\nu)}(s)$ by the point-wise limits;

$$L_0^{(\nu)}(s,\chi) := \lim_{q\downarrow 0} L_q^{(\nu)}(s,\chi) \quad \text{and} \quad \zeta_0^{(\nu)}(s) := \lim_{q\downarrow 0} \zeta_q^{(\nu)}(s).$$

We call the function $\zeta_0^{(\nu)}(s)$ (resp. $L_0^{(\nu)}(s,\chi)$) a *crystal Riemann zeta* (resp. *L*-) *function of type* ν. Note that, in particular, since $\zeta_q^{(\nu)}(s)$ has simple poles at $s = 1, 2, \ldots, \nu$, $\zeta_0^{(\nu)}(s)$ can not be defined at these points. The following proposition is easily obtained from Proposition 2.9.

Proposition 2.10. *Let m be a non-negative integer. Then we have*

$$L_0^{(\nu)}(-m,\chi) = 0 \quad (\chi \neq \mathbf{1}) \quad \text{and} \quad \zeta_0^{(\nu)}(-m) = \begin{cases} -1 & (m=0), \\ 0 & (m \neq 0). \end{cases} \tag{2.25}$$

Further, let $D_0^{(\nu)} := \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) \notin \mathbb{Z}_{\le \nu}\} \cup \{0,-1,-2,\ldots\}$. Then for $s \in D_0^{(\nu)}$, it holds that

$$L_0^{(\nu)}(s,\chi) = 0 \qquad (\chi \neq \mathbf{1}),$$

$$\zeta_0^{(\nu)}(s) = \begin{cases} 0 & \text{if} \quad \mathrm{Re}(s) > \nu, \\ -(s+1)_m/m! & \text{if} \quad \nu - m - 1 < \mathrm{Re}(s) < \nu - m \quad (m = 0,1,2,\ldots). \end{cases}$$

Note that $\lim_{q\downarrow 0} L_q^{(\nu)}(s,\chi)$ does not exist if $s \notin D_0^{(\nu)}$. □

As a corollary of Proposition 2.10, we now determine the zeros of $\zeta_0^{(\nu)}(s)$. This can be regarded as analogues of the Riemann hypothesis for the crystal Riemann zeta functions.

Corollary 2.11. *If $\zeta_0^{(\nu)}(s) = 0$, $\mathrm{Re}(s) \le \nu$ and $s \neq 1, 2, \ldots, \nu$, then $s \in \mathbb{Z}_{<0}$.* □

By this corollary, it seems that the crystal zeta functions have only "trivial" zeros (See the figures in §3). In general, however, no reason can judge which zeros are the trivial or the non-trivial zeros of $\zeta_q^{(\nu)}(s)$ a priori because we do not have a functional equation of $\zeta_q^{(\nu)}(s)$. In the next section, we therefore study numerically the zeros of $\zeta_q^{(\nu)}(s)$ as q-trajectories of the trivial and non-trivial zeros of $\zeta(s)$, respectively, when q approaches 0 decreasing from 1.

Remark 2.12. Let χ be a even character, that is, the equation $\sum_{k=1}^{N} k\chi(k) = 0$ holds. Then $L(1,\chi)$ is expressed as $L'(0,\chi)$ by the functional equation of $L(s,\chi)$. In our case,

$$L_q^{(\nu)\prime}(0,\chi) = \sum_{k=1}^{N} \chi(k) q^{-k\nu} \left\{ \frac{\log q (k + (N-k) q^{-N\nu})}{(1-q^{-N\nu})^2} + \frac{\log(1-q)}{1-q^{-N\nu}} + \sum_{1 \le r \ne \nu} \frac{1}{r} \frac{q^{kr}}{1-q^{N(r-\nu)}} \right\},$$

$$L_q^{(\nu)}(1,\chi) = (1-q) \sum_{k=1}^{N} \chi(k) q^{-k\nu} \left\{ \sum_{1 \le r \ne \nu} \frac{q^{kr}}{1-q^{N(r-\nu)}} \right\}.$$

Hence it is difficult to express $L_q^{(\nu)}(1,\chi)$ in terms of $L_q^{(\nu)\prime}(0,\chi)$. This partially suggests that one may not expect the presence of a functional equation of $L_q^{(\nu)}(s,\chi)$.

Remark 2.13. Let χ be a Dirichlet character modulo N but the principal character. Then the Dirichlet class number formula is given by

$$L(1,\chi) = -\frac{1}{N} \sum_{k=1}^{N} \chi(k) \frac{\Gamma'}{\Gamma}(\frac{k}{N}). \tag{2.26}$$

In our case, similarly, $L_q^{(\nu)}(1,\chi)$ can be expressed in terms of $\Gamma_q(s)$ as

$$L_q^{(\nu)}(1,\chi) = (1-q) \sum_{r=1}^{\nu-1} g_\chi(q^{-\nu+r}) + \frac{1-q}{N \log q} \sum_{k=1}^{N} \chi(k) \frac{\Gamma'_{q^N}}{\Gamma_{q^N}}(\frac{k}{N}). \tag{2.27}$$

Here $\Gamma_q(s)$ is the Jackson q-gamma function defined by $\Gamma_q(s) := \frac{(q;q)_\infty}{(q^s;q)_\infty}(1-q)^{1-s}$, where $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ (see, [Andrews et al. 99]). This is shown by comparing $L_q^{(\nu)}(1,\chi)$ with the logarithmic derivative of $\Gamma_q(s)$. Letting $q \uparrow 1$ in (2.27), we obtain (2.26) by Theorem 2.6. Though a certain quantum analogue of the Dirichlet class number formula was studied in [Kurokawa and Wakayama 02], we remark that the q-analogue of the Hurwitz zeta function defined there is not the same one discussed here. See also [Kurokawa and Wakayama 03].

3 Behavior of zeros of $\zeta_q^{(\nu)}(s)$

In this section, we study the zeros of $\zeta_q^{(\nu)}(s)$ numerically with Maple 8 [Maple 03] and give some conjectures concerning the q-trajectory of the zeros of $\zeta(s)$.

We first notice the

Proposition 3.1. *For all $q \in (0,1]$, $\zeta_q^{(\nu)}(s) \ne 0$ if $\mathrm{Re}(s) \ge 2\nu$.*

Proof. Since $\zeta_q^{(\nu)}(s) = q^{s-\nu} + \sum_{n=2}^{\infty} q^{n(s-\nu)}[n]_q^{-s}$, it is sufficient to show $\sum_{n=2}^{\infty} |q^{n(s-\nu)}[n]_q^{-s}| < |q^{s-\nu}|$. To see this, since $[n]_q = 1 + q + \cdots + q^{n-1} \ge n(1 \cdot q \cdot \cdots \cdot q^{n-1})^{\frac{1}{n}} = nq^{\frac{n-1}{2}}$, it is enough to verify $\sum_{n=2}^{\infty} q^{(n-1)(\frac{\sigma}{2}-\nu)} n^{-\sigma} < 1$, where $\sigma = \mathrm{Re}(s)$. This is actually true because

$$\sum_{n=2}^{\infty} q^{(n-1)(\frac{\sigma}{2}-\nu)} n^{-\sigma} \le \sum_{n=2}^{\infty} n^{-\sigma} < \int_1^{\infty} x^{-\sigma} dx < \frac{1}{\sigma-1} \le 1$$

for $\sigma \ge 2\nu$. This shows the assertion. $\square$

We begin with a study of an approximate formula of $\zeta_q^{(\nu)}(s)$. Put $f(x) = q^{xt}(1-q^x)^{-s}$. Then by the Euler-Maclaurin formula (2.1) again, we have

$$
\begin{aligned}
f_q(s,t) = (1-q)^s \Bigg\{ \sum_{m=1}^{N} q^{mt}(1-q^m)^{-s} - \frac{1}{2} q^{Nt}(1-q^N)^{-s} - \frac{1}{\log q} b_{q^N}(t, -s+1) \\
- \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(N) \Bigg\} - \frac{(1-q)^s}{(2n)!} \int_N^\infty \widetilde{B}_{2n}(x) f^{(2n)}(x)dx.
\end{aligned} \tag{3.1}
$$

Next, we try to evaluate the integral in terms of the incomplete beta functions. For a non-negative integer j, we define the function $h_j(x)$ by $q^{(t+j)x}(1-q^x)^{-s-j}$. Then $h_0^{(n)}(x)(= f^{(n)}(x))$ is expressed by a linear combination of $h_j(x)$'s with coefficients

$$
a_j^{(n)} := (s)_j \sum_{\substack{i_0+i_1+\cdots+i_j \\ =n-j}} t^{i_0}(t+1)^{i_1} \cdots (t+j)^{i_j}.
$$

Lemma 3.2. *For any $n \geq 0$, we have*

$$
h_0^{(n)}(x) = (\log q)^n \sum_{j=0}^{n} a_j^{(n)} h_j(x).
$$

Proof. The diagram (based on Leibniz rule) below describes the rule how the coefficient of $h_j(x)$ in $h_0^{(n)}(x) = \left(q^{tx}(1-q^x)^{-s}\right)^{(n)}$ can be obtained.

$$
\begin{tikzcd}
& & & h_0 \arrow[dl, "t"'] \arrow[dr, "s"] & & & \\
& & h_0 \arrow[dl, "t"'] \arrow[dr, "s"] & & h_1 \arrow[dl, "t+1"'] \arrow[dr, "s+1"] & & \\
& h_0 \arrow[dl, "t"'] \arrow[dr, "s"] & & h_1 \arrow[dl, "t+1"'] \arrow[dr, "s+1"] & & h_2 \arrow[dl, "t+2"'] \arrow[dr, "s+2"] & \\
h_0 & & h_1 & & h_2 & & h_3
\end{tikzcd}
$$

Hence the claim follows immediately. $\square$

By (2.3), (2.5) again and Lemma 3.2, for any integer $M \geq 2$ and $\mathrm{Re}(t) > 1 - M$, the last term of (3.1) containing the integral can be evaluated as

$$
\begin{aligned}
&(1-q)^s (\log q)^{2n-1} \sum_{\substack{l_0 \leq l \leq l_1 \\ l \neq 0}} \sum_{j=0}^{2n} \sum_{k=1}^{M-1} \frac{a_j^{(2n)}}{(2\pi i l)^{2n}} \frac{(-1)^{k-1}(s+j)_{k-1}}{(t+j+\delta l)_k} \frac{q^{N(t+j+k-1)}}{(1-q^N)^{s+j-1+k}} \\
&+ R(s,t,q,N,M,n,l_0,l_1).
\end{aligned}
$$

Here l_0 and l_1 are integers satisfying $l_0 < l_1$, and $R(s,t,q,N,M,n,l_0,l_1)$ is equal to

$$(\log q)^{2n-1}(1-q)^s\Bigg\{\sum_{\substack{l<l_0,l>l_1\\ l\neq 0}}\sum_{j=0}^{2n}\sum_{k=1}^{M-1}\frac{a_j^{(2n)}}{(2\pi i l)^{2n}}\frac{(-1)^k(s+j)_{k-1}}{(t+j+\delta l)_k}\frac{q^{N(t+j+k-1)}}{(1-q^N)^{s+j-1+k}} \\ +\sum_{l\in\mathbb{Z}\setminus\{0\}}\sum_{j=0}^{2n}\frac{a_j^{(2n)}}{(2\pi i l)^{2n}}\frac{(-1)^M(s+j)_{M-1}}{(t+j+\delta l)_{M-1}}b_{q^N}(t+j+\delta l+M-1,-s-j-M+2)\Bigg\}. \tag{3.2}$$

Substituting $s-\nu$ for t in $f_q(s,t)$, we obtain the following expression which allows us to calculate the zeros of $\zeta_q^{(\nu)}(s)$ numerically.

Proposition 3.3. *For integers $N\geq 1, M\geq 2, n\geq 1$ and l_0, l_1 satisfying $l_0 < l_1$, we have for* $\mathrm{Re}(s) > \nu+1-M$

$$\zeta_q^{(\nu)}(s) = (1-q)^s\Bigg\{\sum_{m=1}^{N} q^{m(s-\nu)}(1-q^m)^{-s} - \frac{1}{2}q^{N(s-\nu)}(1-q^N)^{-s} \\ + \frac{1}{\log q}\sum_{r=0}^{\nu-1}\frac{(-\nu+1)_r}{(-s+1)_{r+1}}q^{N(s-\nu)}(1-q^N)^{-s+1+r} - \sum_{k=1}^{n}\sum_{j=0}^{2k-1}\frac{B_{2k}}{(2k)!}(\log q)^{2k-1}a_j^{(2k-1)}q^{N(s-\nu+j)}(1-q^N)^{-s-j} \\ +(\log q)^{2n-1}\sum_{\substack{l_0\leq l\leq l_1\\ l\neq 0}}\sum_{j=0}^{2n}\sum_{k=1}^{M-1}\frac{a_j^{(2n)}}{(2\pi i l)^{2n}}\frac{(-1)^k(s+j)_{k-1}}{(s-\nu+j+\delta l)_k}\frac{q^{N(s-\nu-1+j+k)}}{(1-q^N)^{s+j-1+k}}\Bigg\} \\ + R(s,s-\nu,q,N,M,n,l_0,l_1).$$

Moreover, the absolute value of $R(s,s-\nu,q,N,M,n,l_0,l_1)$ is bounded by

$$\frac{|\log q|^{2n-1}(1-q)^{\mathrm{Re}(s)}}{(2\pi)^{2n}}\Bigg\{\sum_{\substack{l<l_0,l>l_1\\ l\neq 0}}\sum_{j=0}^{2n}\sum_{k=1}^{M-1}\frac{|a_j^{(2n)}|}{l^{2n}}\frac{|(s+j)_{k-1}|}{|(s-\nu+j)_k|}\frac{q^{N(\mathrm{Re}(s)-\nu-1+j+k)}}{(1-q^N)^{\mathrm{Re}(s)+j-1+k}} \\ -\sum_{l\in\mathbb{Z}\setminus\{0\}}\sum_{j=0}^{2n}\frac{|a_j^{(2n)}||(s+j)_{M-1}|}{l^{2n}|(\mathrm{Re}(s)-\nu+j)_{M-1}|}\sum_{r=0}^{\nu-1}\frac{(-\nu+1)_r}{(-\mathrm{Re}(s)-j-M+2)_{r+1}}\frac{q^{\mathrm{Re}(s)+j+M-\nu-1}}{(1-q)^{\mathrm{Re}(s)+j+M-2-r}}\Bigg\}.$$

Proof. Let $t=s-\nu$. Then the first assertion follows immediately from (3.1). The second assertion follows from (3.2) and (2.9). This completes the proof. $\square$

If s is real, then $\zeta_q^{(\nu)}(s)$ is also real by the definition of $\zeta_q^{(\nu)}(s)$. We actually compute approximate values of the zeros of $\zeta_q^{(\nu)}(s)$ by searching the point where the sign of $\zeta_q^{(\nu)}(s)$ changes. If s is complex, we can not, however, determine the place of the zeros of $\zeta_q^{(\nu)}(s)$ as we can not expect an existence of a functional equation of $\zeta_q^{(\nu)}(s)$. Thus we try to seek the points s_0 which are expected as the zeros of $\zeta_q^{(\nu)}(s)$ by observing respectively the sign of the real part and the imaginary part of $\zeta_q^{(\nu)}(s)$. Indeed, if $\zeta_q^{(\nu)}(s_0)=0$, then the signs of both $\mathrm{Re}(\zeta_q^{(\nu)}(s))$ and $\mathrm{Im}(\zeta_q^{(\nu)}(s))$ will change in a neighborhood of the point s_0, and consequently, $|\zeta_q^{(\nu)}(s)|$ is very small in the neighborhood of s_0. We find a certain neighborhood in which the both of the signs simultaneously change.

Following the strategy mentioned above, we examine the sign of $\zeta_q^{(\nu)}(s)$ when s is real, and the sign of $\mathrm{Re}(\zeta_q^{(\nu)}(s))$ and of $\mathrm{Im}(\zeta_q^{(\nu)}(s))$ when s is complex. In these numerical calculations, the parameter q moves from 0.99 to 0.01 by 0.01, and additionally 3 points 0.001, 0.0001, and 0.00001. The parameter s moves from the real zero point at $q = 1$ by 0.001 on the points of the real axis if s is real, and on the points of the lattice parallel to the real and imaginary axis if s is complex. Furthermore we may carefully select parameters q, N, M, n, l_0, l_1 satisfying $|R(s, s-\nu, q, N, M, n, l_0, l_1)| < 10^{-5}$ on each point s. In the following figures, the point $\times$ represents the zeros of $\zeta(s)$. When $q = 0.99, \ldots, 0.01$, we plot approximate (if s is real) or expecting points (if s is complex) by $\bullet$, and when q is the other 3 points, by $\star$. Further, $\bullet$, $\bullet$ and $\bullet$ denote $\nu = 1, 2$ and 3, respectively.

1. Trivial zero's cases:
First we observe that the behavior of the zeros of $\zeta_q^{(\nu)}(s)$ $(q \in (0, 1])$ starting from the trivial zeros of $\zeta(s)$. Put $s_j = -2j$, the j-th trivial zero of $\zeta(s)$. Let $s_j^{(\nu)}(q)$ be the q-trajectory of $s_j = s_j^{(\nu)}(1)$ satisfying $\zeta_q^{(\nu)}(s_j^{(\nu)}(q)) = 0$. We plot $s_j^{(\nu)}(q)$ approximately on the (s, q)-plane. Note that $s_j^{(\nu)}(q)$ is real. Figure 1, 2 show the q-trajectory of $s_1^{(\nu)}(q)$, $s_2^{(\nu)}(q)$ respectively. It can be expected from these two figures that $s_1^{(\nu)}(q)$, $s_2^{(\nu)}(q)$ approach $s = -1, -2$ respectively for *all* $\nu = 1, 2, 3$. Moreover, by virtue of further numerical calculations, it seems true that $s_3^{(\nu)}(q)$, $s_4^{(\nu)}(q)$ approach $s = -3$, -4 respectively for *all* $\nu = 1, 2, 3$. This motivates the

Conjecture. *The limit* $s_j^{(\nu)}(0) := \lim_{q \downarrow 0} s_j^{(\nu)}(q)$ *exists and is given by* $s_j^{(\nu)}(0) = s_j/2 = -j$ *for all* $\nu \in \mathbb{N}$. □

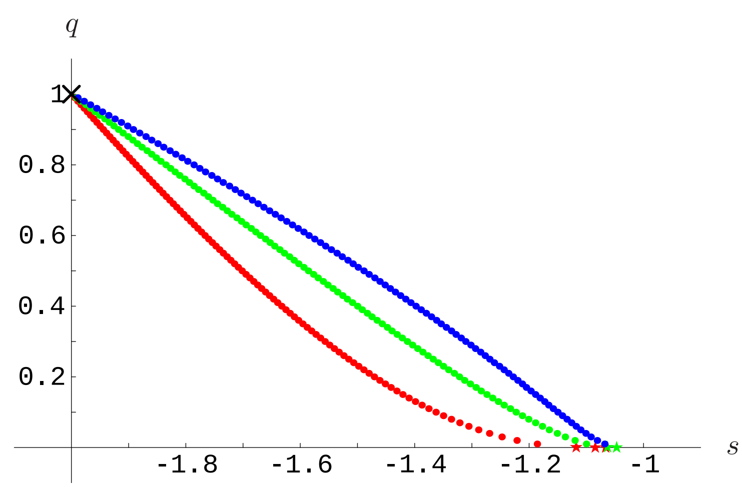

Figure 1: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $s_1 = -2$

The next proposition may partially support this conjecture.

Proposition 3.4. *Let* $z^{(\nu)}(q)$ *be a zero of* $\zeta_q^{(\nu)}(s)$. *If* $\frac{\partial}{\partial s}\zeta_q^{(\nu)}(z^{(\nu)}(q)) \neq 0$ *and* $z^{(\nu)}(q)$ *is right continuous at* $q = 0$ *and* $z^{(\nu)}(0) := \lim_{q \downarrow 0} z^{(\nu)}(q) = -m$ *for an integer* m, *then* $\lim_{q \downarrow 0} \frac{d}{dq} z^{(\nu)}(q) = \pm\infty$. *In particular,* $z^{(\nu)}(q)$ *is tangent to the line* $q = 0$ *at* $s = -m$ *in* $(s, q) \in \mathbb{R} \times [0, 1]$.

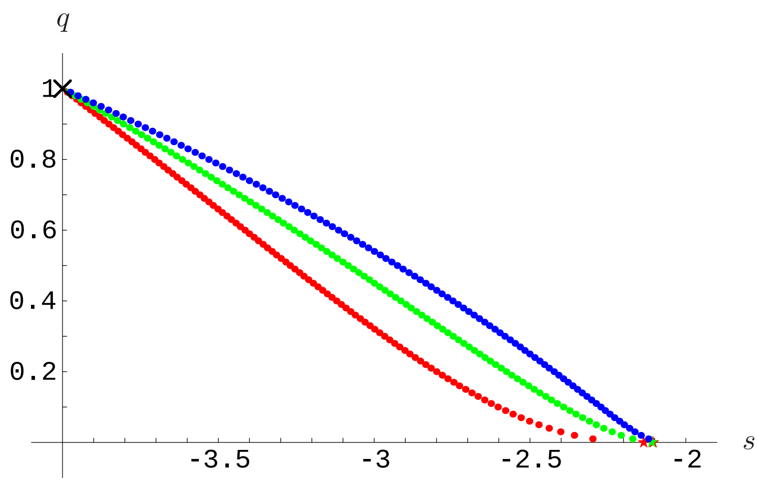

Figure 2: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $s_2 = -4$

Proof. Note that $\zeta_q^{(\nu)}(s)$ is real for $s \in \mathbb{R}$. Since $\frac{\partial}{\partial s}\zeta_q^{(\nu)}(z^{(\nu)}(q)) \neq 0$, we conclude that $z^{(\nu)}(q)$ is differentiable near $q = 0$ and $\frac{d}{dq}z^{(\nu)}(q) = -\frac{\partial}{\partial q}\zeta_q^{(\nu)}(z^{(\nu)}(q))\frac{\partial}{\partial s}\zeta_q^{(\nu)}(z^{(\nu)}(q))^{-1}$ by the implicit function theorem. Since, by the expression (2.21), $\frac{\partial}{\partial q}\zeta_q^{(\nu)}(s)$ and $\frac{\partial}{\partial s}\zeta_q^{(\nu)}(s)$ can be written as

$$
\begin{aligned}
\frac{\partial}{\partial q}\zeta_q^{(\nu)}(s) &= -\frac{s}{1-q}\zeta_q^{(\nu)}(s) + q^{-1}(1-q)^{-m} \\
&\quad \times \left\{ \frac{(-1)^m m!(\nu-1)!}{(\nu+m)!(\log q)^2} + \sum_{\substack{r=0 \\ r\neq \nu+m}}^{\infty} \binom{-m+r-1}{r} \frac{(-m+r-\nu)q^{-m+r-\nu}}{(1-q^{-m+r-\nu})^2} \right\} + h_1(s), \\
\frac{\partial}{\partial s}\zeta_q^{(\nu)}(s) &= \log(1-q)\zeta_q^{(\nu)}(s) + (1-q)^{-m} \left\{ \frac{(-1)^{m+1} m!(\nu-1)!}{(\nu+m)!\log q} \Big(\sum_{\substack{k=0 \\ k\neq m}}^{m+\nu-1} \frac{1}{-m+k} + \frac{1}{2}\log q \Big) \right. \\
&\quad + \sum_{r=0}^{m} \binom{-m+r-1}{r} \frac{q^{-m+r-\nu}}{1-q^{-m+r-\nu}} \Big(\sum_{k=0}^{r-1} \frac{1}{-m+k} + \frac{\log q}{1-q^{-m+r-\nu}} \Big) \\
&\quad \left. + \sum_{\substack{r=m+1 \\ r\neq m+\nu}}^{\infty} \frac{(-1)^m m!(r-m-1)!}{r!} \frac{q^{-m+r-\nu}}{1-q^{-m+r-\nu}} \right\} + h_2(s),
\end{aligned}
$$

where $h_i(s)$ is a function satisfying $h_i(-m) = 0$ $(i = 1, 2)$, we have

$$
\begin{aligned}
\lim_{q\downarrow 0} \frac{\partial}{\partial q}\zeta_q^{(\nu)}(z^{(\nu)}(q)) &= \pm\infty, \\
\lim_{q\downarrow 0} \frac{\partial}{\partial s}\zeta_q^{(\nu)}(z^{(\nu)}(q)) &= \frac{(-1)^{m+1} m!(\nu-1)!}{2(\nu+m)!} - \sum_{r=0}^{m} \binom{-m+r-1}{r} \sum_{k=0}^{r-1} \frac{1}{-m+k} \\
&\quad - \sum_{r=m+1}^{m+\nu-1} \frac{(-1)^{m+1} m!(r-m-1)!}{r!}.
\end{aligned}
$$

This shows the assertion. $\square$

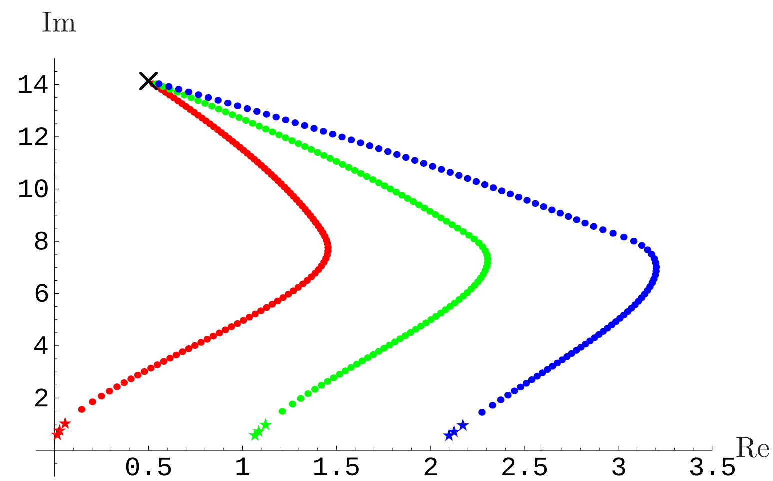

Figure 3: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $\rho_1 = \frac{1}{2} + \sqrt{-1} \times 14.13472\ldots$

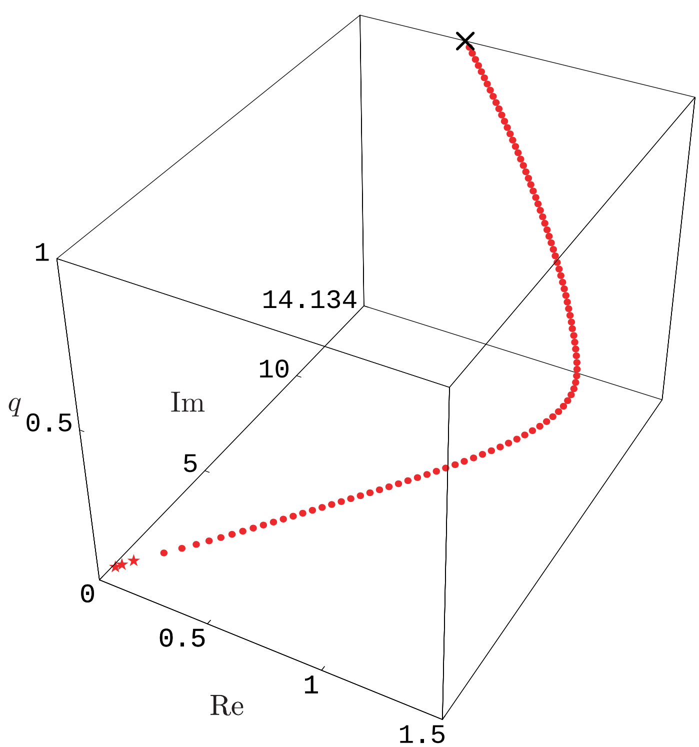

Figure 4: 3-dimensional graph of Figure 3 for $\nu = 1$

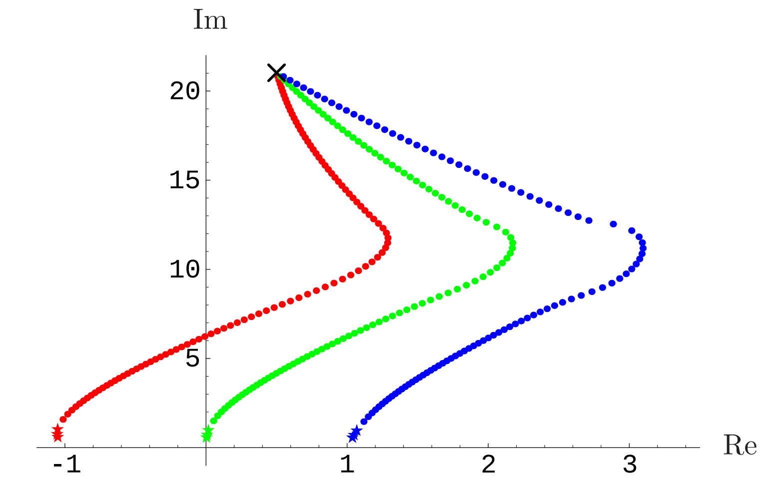

Figure 5: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $\rho_2 = \frac{1}{2} + \sqrt{-1} \times 21.02203\ldots$

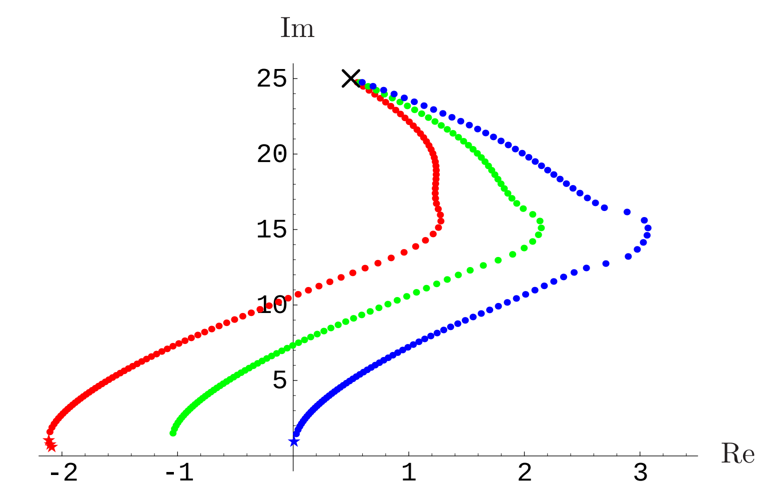

Figure 6: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $\rho_3 = \frac{1}{2} + \sqrt{-1} \times 25.01085\ldots$

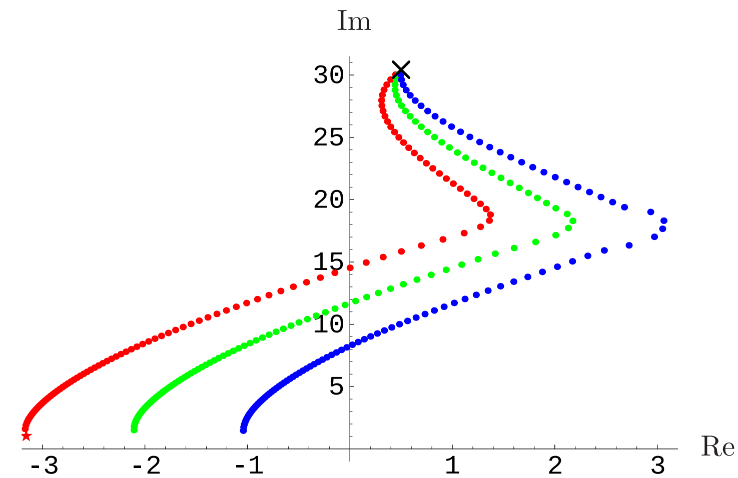

Figure 7: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $\rho_4 = \frac{1}{2} + \sqrt{-1} \times 30.42487\ldots$

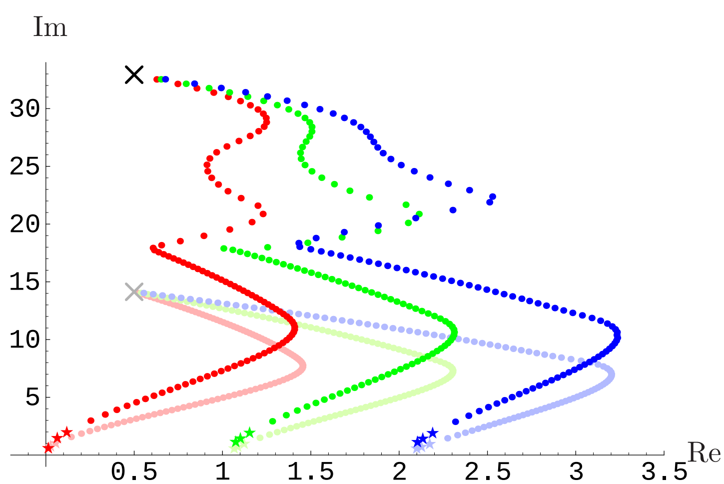

Figure 8: the trajectory of zeros of $\zeta_q^{(\nu)}(s)$ starting from $\rho_5 = \frac{1}{2} + \sqrt{-1} \times 32.93506\ldots$ with ρ_1-trajectory (see Figure 3)

2. Non-trivial zero's cases:
We here investigate the behavior of the zeros of $\zeta_q^{(\nu)}(s)$ $(q \in (0,1])$ starting from the non-trivial zeros of $\zeta(s)$. Let ρ_j $(j = 1, 2, \dots)$ be the j-th non-trivial zero of $\zeta(s)$ whose imaginary part is positive, that is, $\rho_1 = \frac{1}{2} + \sqrt{-1} \times 14.13472\dots$, $\rho_2 = \frac{1}{2} + \sqrt{-1} \times 21.02203\dots$, $\rho_3 = \frac{1}{2} + \sqrt{-1} \times 25.01085\dots$, ... Let $\rho_j^{(\nu)}(q)$ be the q-trajectory of $\rho_j = \rho_j^{(\nu)}(1)$ satisfying $\zeta_q^{(\nu)}(\rho_j^{(\nu)}(q)) = 0$. Note that, by the principle of reflection, we have $\zeta_q^{(\nu)}(\overline{\rho_j^{(\nu)}(q)}) = 0$. Figure 3 (see also Figure 4), Figure 5, Figure 6, Figure 7 and Figure 8 correspond to the case ρ_1, ρ_2, ρ_3, ρ_4 and ρ_5 respectively. In these cases, we plot $\rho_j^{(\nu)}(q)$ on the complex plane. $\rho_j^{(\nu)}(q)$ $(j = 1, 2, \dots, 5)$ seem to arrive at 0, -1, -2, -3 and 0 respectively for $\nu = 1$ when $q \downarrow 0$. It is easy to see that if ν increases 1, the real part of $\rho_j^{(\nu)}(q)$ also increases 1 near $q = 0$.

Conjecture. *The limit* $\rho_j^{(\nu)}(0) := \lim_{q\downarrow 0} \rho_j^{(\nu)}(q)$ *exists and satisfies* $\rho_j^{(\nu)}(0) \in \mathbb{Z}_{\leq \nu-1}$. *Additionally, the relation* $\rho_j^{(\nu+1)}(0) = \rho_j^{(\nu)}(0) + 1$ *holds for* $\nu \in \mathbb{N}$. □

Remark 3.5. If $\rho_j^{(\nu)}(q)$ is right continuous at $q = 0$, then we can show that $\mathrm{Re}(\rho_j^{(\nu)}(0)) \in \mathbb{Z}_{<0}$. In fact, suppose otherwise. Then there exists some positive integer N such that $-N < \mathrm{Re}(\rho_j^{(\nu)}(q)) < -N + 1$ for $q \in [0, \varepsilon)$ for some $\varepsilon > 0$. Since $\zeta_q^{(\nu)}(\rho_j^{(\nu)}(q)) = 0$, the continuity of $\rho_j^{(\nu)}(q)$ and the existence of the limit show that $0 = \lim_{q\downarrow 0} \zeta_q^{(\nu)}(\rho_j^{(\nu)}(q)) = \zeta_0^{(\nu)}(\rho_j^{(\nu)}(0))$. This contradicts the results in Proposition 2.10.

Remark 3.6. Remark that the limit behavior of the 5th zeros (Figure 8) is different from the one what can be expected from the 1st, 2nd, 3rd and 4th zeros. See also Figures 13, 14, 15, and the subsequent observation.

3. Other remarks and some questions:
We give another numerical calculation of $\zeta_q(s) = \zeta_q^{(1)}(s)$ at the neighborhoods of $s = 0, -1$ with small q from the viewpoint of the approximations obtained by Proposition 2.9:

$$\zeta_q(s) \approx \begin{cases} (1-q)^s\left(\frac{q^{s-1}}{1-q^{s-1}} + s\frac{q^s}{1-q^s}\right) & \text{if } \mathrm{Re}(s) > -1, \\ (1-q)^s\left(\frac{q^{s-1}}{1-q^{s-1}} + s\frac{q^s}{1-q^s} + \frac{s(s+1)}{2}\frac{q^{s+1}}{1-q^{s+1}}\right) & \text{if } \mathrm{Re}(s) > -2. \end{cases}$$

Figure 9 and Figure 10 indicate the logarithm of absolute values of these forms on the rectangles $[-0.05, 0.05] \times [0, 1]$ and $[-1.05, -0.95] \times [0, 1]$ with $q = 2^{-64}$, respectively. The horizontal line represents the imaginary axis in Figure 9 and the line $\mathrm{Re}(s) = -1$ in Figure 10 respectively. The vertical one indicates the real axis. The variation of the logarithmic scale is represented by gray scale. If the absolute value is larger than 1.5, the color of the point is white, and the color of the point at which the absolute value is the smallest (locally) is black. For the sake of a proper understanding, we give also 3-dimensional graphs for Figure 9 and 10 in Figure 11 and 12 respectively. Notice that the height in these 3-dimensional figures is taken by an absolute value, not the logarithm of it. We can see that the most left black holes of Figure 9 corresponds to the trajectory of the non-trivial zero $\rho_1 = \frac{1}{2} + \sqrt{-1} \times 14.13472\dots$ and the most left large hole soaked

with black and the second black hole of Figure 10 correspond to the trajectories of the trivial zero $s_1 = -2$ and of $\rho_2 = \frac{1}{2} + \sqrt{-1} \times 21.02203\ldots$ by looking at the consequences of numerical calculations performing from $q = 2^{-13} \approx 1.2 \times 10^{-4}$ to $q = 2^{-64}$. Also, the second black hole in Figure 9 seems to represent the point in the trajectory of ρ_5 (see Figure 8 and Remark 3.6). It seems that there are 6 and 7 black holes which may indicate the zeros of $\zeta_q(s)$ and these black holes lie on a line parallel to the imaginary axis. This comes from the periodicity of the function $\frac{q^{s+r-1}}{1-q^{s+r-1}}$, that is, $|\zeta_q(s+\delta n)|$ is very small for $n = \pm 1, \pm 2, \ldots$ provided $\zeta_q(s) = 0$. Actually, if q is very small, then $\binom{s+\delta n+r-1}{r} \approx \binom{s+r-1}{r}$ and $(1-q)^{\delta n} \approx 1$. Therefore, by Proposition 2.9, we have $\zeta_q(s+\delta n) = (1-q)^{s+\delta n} \sum_{r=0}^{\infty} \binom{s+\delta n+r-1}{r} \frac{q^{s+\delta n+r-1}}{1-q^{s+\delta n+r-1}} \approx \zeta_q(s) = 0$. Thus, when $q = 2^{-64}$, i.e. $2\pi/\log q = -0.1416\ldots$, such a black hole appears approximately every 0.14 along the horizontal lines in Figure 9 and Figure 10 respectively.

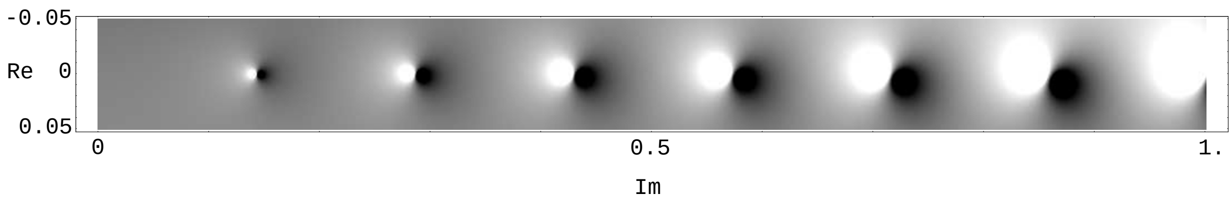

Figure 9: $|\zeta_q(s)|$ around $s = 0$ at $q = 2^{-64} \approx 5 \times 10^{-20}$

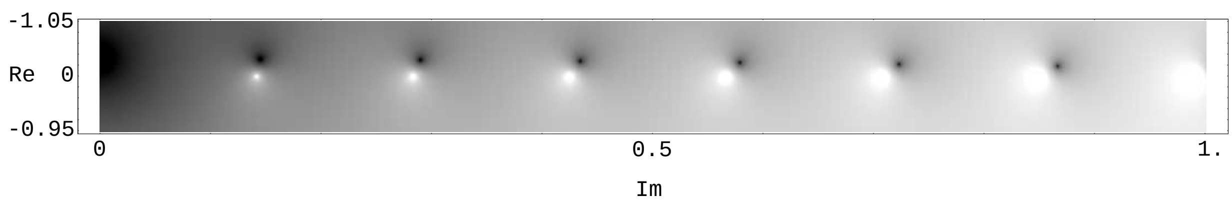

Figure 10: $|\zeta_q(s)|$ around $s = -1$ at $q = 2^{-64} \approx 5 \times 10^{-20}$

Proposition 3.4 indicates that the trajectory $z^{(\nu)}(q)$ can be tangent to the real axis when q approaches to zero, but it is difficult to determine the approaching direction theoretically. From Figure 5, 6 and 7, however, we can verify that the one of the points $\rho_2^{(1)}(q)$, $\rho_3^{(1)}(q)$, $\rho_3^{(2)}(q)$ and $\rho_4^{(\nu)}(q)$ $(\nu = 1, 2, 3)$ approaching a negative integer $-m$ seems to be tangent to the real axis from the left direction, while $\rho_1^{(\nu)}(q)$ $(\nu = 1, 2, 3)$, $\rho_2^{(2)}(q)$, $\rho_2^{(3)}(q)$, $\rho_3^{(3)}(q)$ and $\rho_5^{(\nu)}(q)$ $(\nu = 1, 2, 3)$ are not the cases. For instance, it seems that $\rho_3^{(1)}(q)$ crosses the line $\mathrm{Re}(s) = -m$ first, then turns to the right and goes to $-m$ when q approaches to 0. This is consistent with the periodicity one can see in Figure 9 and 10 mentioned above.

Lastly, we display the location of zeros $\rho_j^{(1)}(q)$ of $\zeta_q(s) = \zeta_q^{(1)}(s)$ approximately which lie on the trajectory starting from the first seven non-trivial zeros of $\zeta(s)$ when $q = 0.9$, 0.6 and 0.3 respectively in Figures 13-15. In the figures, the number j indicates $\rho_j^{(1)}(q)$ for simplicity.

From the numerical calculation developed above, the following questions naturally come up.

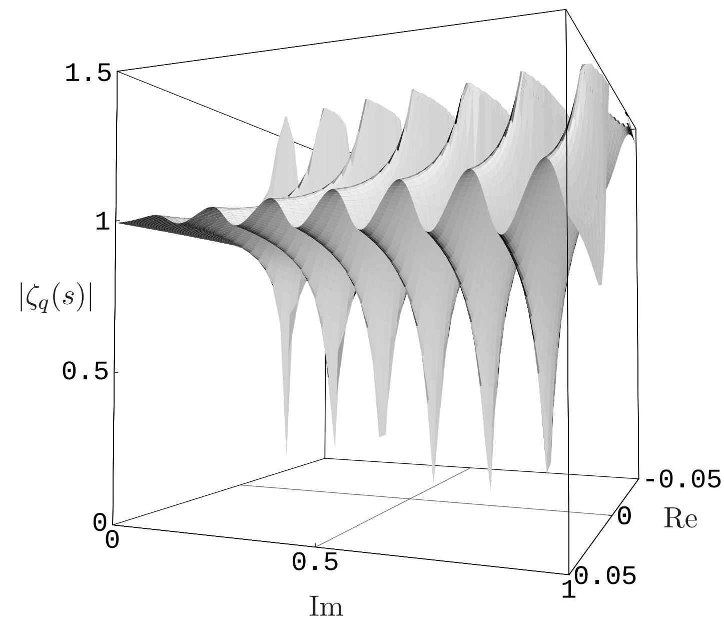

Figure 11: 3-dimensional graph for Figure 9

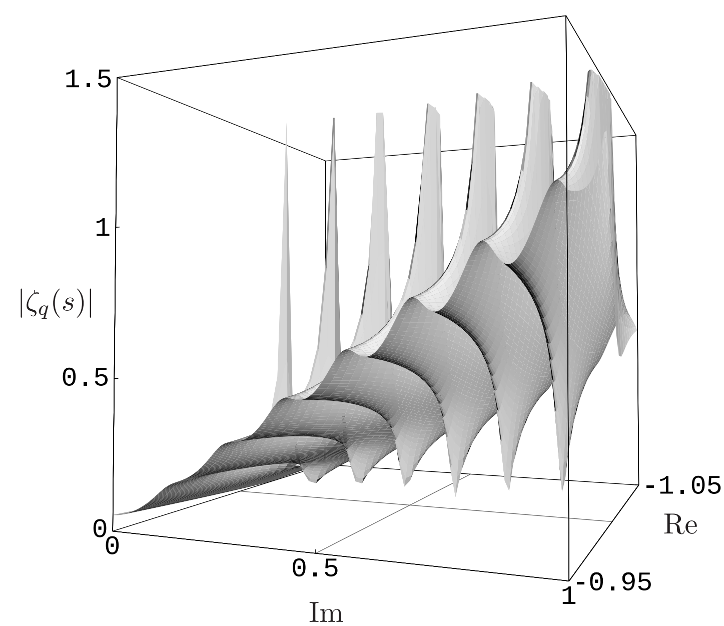

Figure 12: 3-dimensional graph for Figure 10

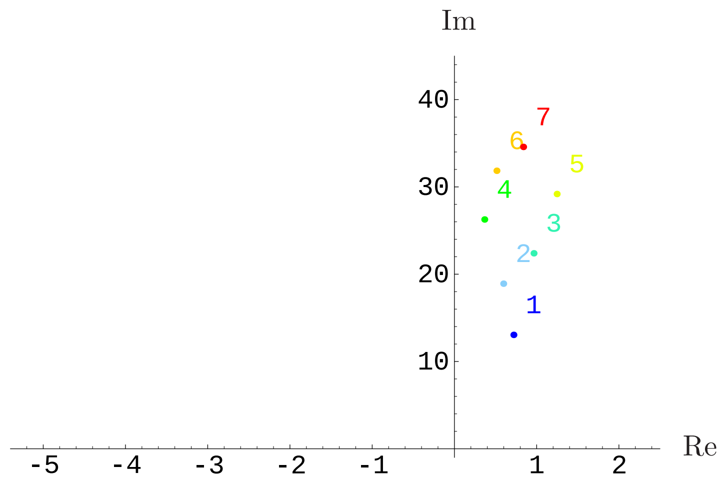

Figure 13: the zeros of $\zeta_q(s)$ correspond to the non-trivial zeros of $\zeta(s)$ when $q = 0.9$

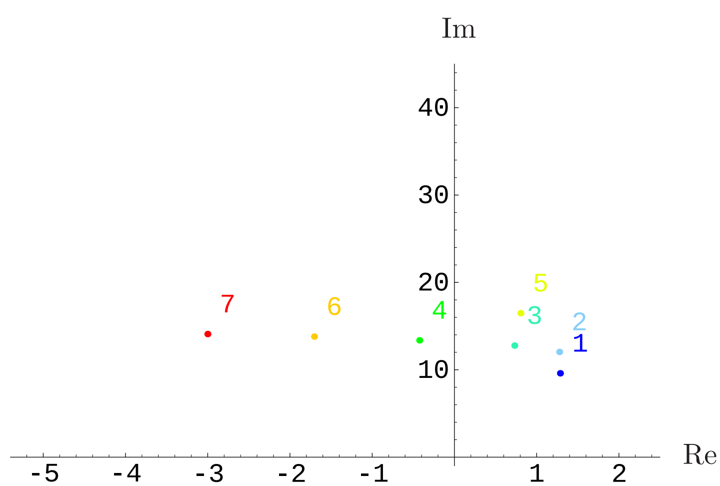

Figure 14: the zeros of $\zeta_q(s)$ correspond to the non-trivial zeros of $\zeta(s)$ when $q = 0.6$

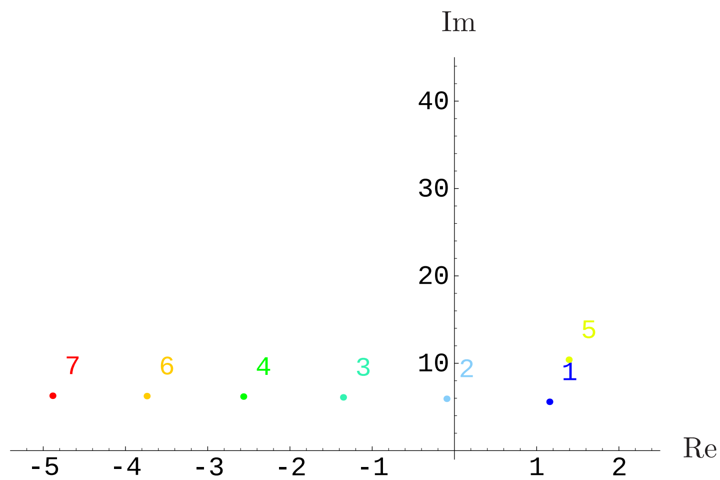

Figure 15: the zeros of $\zeta_q(s)$ correspond to the non-trivial zeros of $\zeta(s)$ when $q = 0.3$

Questions. *(i) Does the relation* $\mathrm{Im}(\rho_j^{(\nu)}(q)) > \mathrm{Im}(\rho_l^{(\nu)}(q))$ $(q \in (0,1))$ *hold for* $j > l$*? In particular, is* $\rho_j^{(\nu)}(q)$ *a simple zero of* $\zeta_q^{(\nu)}(s)$ *whenever* $\rho_j^{(\nu)} = \rho_j^{(\nu)}(1)$ *is simple?*
(ii) What is the shape of the function $\rho_j^{(\nu)}(q)$*? For instance, where is the extremal point of the graph of* $\mathrm{Re}(\rho_j^{(\nu)}(q))$ *as a function of* $\mathrm{Im}(\rho_j^{(\nu)}(q))$*? etc. Moreover, what is an analogue of the Riemann hypothesis for* $\zeta_q^{(\nu)}(s)$*?*

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References

[Andrews et al. 99] G.E. Andrews, R. Askey and R. Roy, Special Functions, Encyclopedia of math. and appl., vol. 71, Cambridge U.P., 1999.

[Cherednik 01] I. Cherednik, On q-analogues of Riemann's zeta function, *Selecta. math.* **7** (2001), 447–491.

[Kaneko et al. 03] M. Kaneko, N. Kurokawa and M. Wakayama, A variation of Euler's approach to values of the Riemann zeta function, *Kyushu J. Math.* **57** (2003), 175–192.

[Kurokawa and Wakayama 02] N. Kurokawa and M. Wakayama, Generalized zeta regularization, quantum class number formulas, and Appell's O-function, to appear in *The Ramanujan J.*

[Kurokawa and Wakayama 03] ________, On q-analogue of the Euler constant and Lerch's limit formula, *Proc. Amer. Math. Soc.* **132** (2003), 935–943.

[Maple 03] MAPLE. Available from World Wide Web : (http://www.maplesoft.com), 2003.

[Satoh 89] J. Satoh, q-Analogue of Riemann's ζ-function and q-Euler numbers, *J. Number Theory.* **31** (1989), 346–362.

[Titchmarsh 86] E.C. Titchmarsh, The Theory of the Riemann Zeta-function, 2nd edn, Oxford Science Publications, 1986.

[Tsumura 01] H. Tsumura, On modification of the q-L-series and its applications, Nagoya Math. J. **164** (2001), 185–197.

[Zhao] J. Zhao, q-Multiple Zeta Functions and q-Multiple Polylogarithms, preprint, 2003. math.QA/0304448

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